Correlation criteria for Bell type inequalities and entanglement detection

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We provide a novel criterion for identifying quantum correlation, which allows us to find connections between Bell type inequalities, entanglement detection, and correlation. We utilize the criterion to construct witness operators that can detect genuine multi-qubit entanglement with fewer local measurements. The connection between identifications of quantum correlation and Mermin's inequality is discussed. Detection of genuine four-level tripartite entanglement with two local measurement settings is shown in the same manner. Further, through the criterion of quantum correlation, we derive a new Bell inequality for arbitrary high-dimensional bipartite systems, which requires fewer analyses of the measured outcomes.

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Introduction.— Bell type inequalities [1, 2, 3, 4] and entanglement witnesses (EW) [6, 7, 8] lie at the heart of entanglement verification for quantum information processing [9]. Recently, the stabilizer formalism has been utilized to derive Bell type inequalities [10] and EW for multi-qubit systems [8]. It has been shown that detections of genuine entanglement (GE) around several types of stabilizer states require only two local measurement settings, and it also has been found that the stabilizer witnesses are closely related to Mermin-type Bell inequalities [8]. However, connections between multilevel Bell type inequalities and EW are still not clear. Besides, there still lakes a general way to detect a genuine multilevel multipartite entanglement with fewer local measurements.

In this work, we present a new type of criterion for identifying quantum correlation (QC), which is helpful for the investigation on the subjects mentioned above. Firstly, EW for detecting genuine multi-qubit entangled states are presented, including detection of entanglement for states close to ones with nonlocal stabilizing operators, e.g., the four-qubit state [11]. Connections between Mermin-type Bell inequalities [3] and criteria of QC are discussed. Secondly, we generalize the utility of correlation criteria and propose the first EW for detecting GE around a four-level tripartite Greenberger-Horne-Zeilinger (GHZ) state with two local measurement settings. Finally, through identifications of QC, we give a new Bell type inequality for arbitrary high-dimensional bipartite systems with fewer analyses of the measured outcomes.

Criteria of QC and detection of GE for qubits.— We first illustrate the main notion of our strategy by providing a EW to detect GE around a four-qubit GHZ state. According to our knowledge of the physical state which is represented in the eigenbasis of the Pauli matrix σ_z: $|\,\mathrm{GHZ}\rangle = (|0000\rangle_z + |1111\rangle_z)/\sqrt{2}$, where $|kkkk\rangle_z \equiv |k\rangle_{1,z} \otimes |k\rangle_{2,z} \otimes |k\rangle_{3,z} \otimes |k\rangle_{4,z}$, we give four sets of correlators to describe the QC between *a specific party* and *others*:

$$C_{0,n} = \sum_{k=0}^{1} (-1)^k P(v_n = k, \mathrm{v} = 0), \qquad (1)$$

$$C_{1,n} = \sum_{k=0}^{1} (-1)^{k+1} P(v_n = k, \mathrm{v} = 3), \qquad (2)$$

where $\mathrm{v} \equiv \sum_{i=1,i\neq n}^{4} v_i$, v_n denotes the outcome of a measurement performed on the n^{th} particle for $n = 1,..,4$, and $P(v_n = k, \sum_{i=1,i\neq n}^{4} v_i = 3m)$ stands for a joint probability for obtaining $v_n = k$ and $v_i = m$ for $m = 0,1$. If results of measurements reveal that $C_{0,n}C_{1,n} > 0$, we are convinced that the outcomes of measurements performed on the n^{th} particle are correlated with the ones performed on the rest [12]. Further, with a *prior* information about probabilities for outcomes of measurements of a GHZ state, $\mathcal{I}_{z,\mathrm{GHZ}}$: $P(v_{nm} = 0) + P(v_{nm} = 2) = 1$, where $v_{nm} \equiv v_n + v_m$; $n, m = 1, ..., 4$, and $n \neq m$, we construct following correlators to identify correlations between *a specific group* , which is composed of the n^{th} party and the m^{th} one, and *another*:

$$C_{0,nm} = \sum_{k=0}^{1} (-1)^k P(v_{nm} = 2k, \mathrm{v}' = 0), \qquad (3)$$

$$C_{1,nm} = \sum_{k=0}^{1} (-1)^{k+1} P(v_{nm} = 2k, \mathrm{v}' = 2), \qquad (4)$$

where $\mathrm{v}' \equiv \sum_{i=1,i\neq n\neq m}^{4} v_i$. It is clear that $C_{0,nm}C_{1,nm} > 0$ for a pure GHZ state, which indicates the subsystem composed of the n^{th} and the m^{th} parts are correlated with another [12].

We consider the sets of correlators $C_{0,n}$, $C_{1,n}$, $C_{0,mn}$, and $C_{1,mn}$ as identifications of a four-qubit GHZ state under the the local measurement setting $\sigma_z^{\otimes 4}$ and take a combination of these correlators: $C^{(z)} = \sum_{n=1}^{4}(C_{0,n} +$

$C_{1,n}) + \sum_{m=2}^{4}(C_{0,1m} + C_{1,1m})$, as one criterion of QC. Moreover, each correlator can be characterized by a formulation of Hermitian operators. For instance, $C_{0,k}$ can be characterized by the operator $\hat{C}_{0,k} = (\hat{0}_k - \hat{1}_k)\hat{0}_i\hat{0}_j\hat{0}_l$ for $k,i,j,l = 1,...,4$ and $k \neq i \neq j \neq l$, where $\hat{p}_q \equiv |p\rangle_{qq}\langle p|$ for $p = 0,1$ and $q = 1,...,4$. Thus, we derive the correlator operator: $\hat{C}^{(\mathrm{I})} = 8(\hat{0}\hat{0}\hat{0}\hat{0} + \hat{1}\hat{1}\hat{1}\hat{1}) - \mathbb{1}$, from $C^{(z)}$.

After introducing the first kind criterion involved $\hat{C}^{(\mathrm{I})}$, let us progress to present the second one for QC. One can acquire the prior information, $\mathcal{I}_{x,\mathrm{GHZ}}$: $\sum_{m=0}^{2} P(\sum_{n=1}^{4} v_n = 2m) = 1$, from the wave function of a four-qubit GHZ state which is represented in the eigenbasis of the Pauli matrix σ_x , i.e., $|\mathrm{GHZ}\rangle = \sum_{m,n,i,j=0}^{1} \delta[(m+n+i+j)\mathrm{mod}2, 0]\,|mnij\rangle_x /2\sqrt{2}$. Thus, through $\mathcal{I}_{x,\mathrm{GHZ}}$, we formulate four sets of criteria which correspond to the following projection operators for identifying the correlations between the k^{th} party and others: $\hat{C}^{(x)}_{0,k} = (\hat{0}_k - \hat{1}_k)(\hat{0}_i\hat{0}_j\hat{0}_l + \hat{0}_i\hat{1}_j\hat{1}_l + \hat{1}_i\hat{0}_j\hat{1}_l + \hat{0}_i\hat{1}_j\hat{1}_l)$, $\hat{C}^{(x)}_{1,k} = (\hat{1}_k - \hat{0}_k)(\hat{1}_i\hat{1}_j\hat{1}_l + \hat{1}_i\hat{0}_j\hat{0}_l + \hat{0}_i\hat{1}_j\hat{0}_l + \hat{0}_i\hat{0}_j\hat{1}_l)$. Since the expectation values of operators, $\hat{C}^{(x)}_{0,k}$ and $\hat{C}^{(x)}_{1,k}$, are all positive for a pure GHZ state, we ensure that there are correlations between outcomes under the measurement setting $\sigma_x^{\otimes 4}$. We combine the four sets of correlators and deduce the Hermitian operators: $\hat{C}^{(\mathrm{II})}_1 = \sum_{k=1}^{4}\sum_{m=0}^{1} \hat{H}^{\otimes 4}\hat{C}^{(x)}_{m,k}\hat{H}^{\otimes 4} = 4X_1X_2X_3X_4$, where $\hat{H}$ is the Hadamard operator [9] and $X_m = \sigma_x$ for the m^{th} party. Similarly, as for identifications of QC between each party and others, we find that the following operators work as well as $\hat{C}^{(\mathrm{II})}_1$: $\hat{C}^{(\mathrm{~II})}_2 = 4Y_1Y_2Y_3Y_4$ where Y_m is the Pauli matrix σ_y; and the set of operators which involve permutations of $XXYY$: $\hat{C}^{(\mathrm{II})}_3 = -4X_1X_2Y_3Y_4$,..., $\hat{C}^{(\mathrm{II})}_8 = -4Y_1Y_2X_3X_4$.

Then, we combine both kinds of criteria of QC as a identification of a four-qubit GHZ state and utilize the the witness operator: $\mathcal{W}_{\mathrm{GHZ}} = \tau\mathbb{1} - (c_1\hat{C}^{(\mathrm{I})} + c_2\hat{C}^{(\mathrm{II})}_k)$, for $k = 1,...,8$, to identify a state ρ as an *genuinely* entangled one which is close to a four-qubit GHZ state if it follows the condition: $\mathrm{Tr}[\mathcal{W}_{\mathrm{GHZ}}\rho] < 0$. The values of τ and $c_{1,2}$ can be determined by the condition of a multi-qubit witness [8]: $\mathcal{W}_{\mathrm{GHZ}} - \gamma\mathcal{W}_{\mathrm{GHZ_p}} \geq 0$ where γ is some positive constant and $\mathcal{W}_{\mathrm{GHZ_p}}$ is the projector-based witness [7], and we have $\tau = 7$, $c_1 = c_2 = 1$, and $\gamma = 8$. Moreover, when a state mixes with white noise, $\rho = p\mathbb{1}/16 + (1-p)\,|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}|$, it is identified as an GE which is close to a GHZ state when $p < 4/11 (\approx 0.363636)$. Through our knowledge of the N-qubit GHZ state and the same way presented above, one can formulate sets of correlators to identify correlations between a group composed of m parties and the rest $(N - m)$ parts. From these criteria, we can construct the witness operator that detects states around a N-qubit GHZ state [15].

Let us proceed to consider a scenario of entanglement detection which involves only the second kind criteria. For the absence of the first kind criterion to identify correlations between two groups, we find that the operator, $\mathcal{W}^{(\mathrm{II})}_2 = \tau\mathbb{1} - (\hat{C}^{(\mathrm{~II})}_k + \hat{C}^{(\mathrm{II})}_{k'})$ for $k' \neq k$, cannot satisfy $\mathcal{W}^{(\mathrm{II})}_2 - \gamma\mathcal{W}_{\mathrm{GHZ~p}} \geq 0$. However, if we add more two terms to the operator, it will be the case. For instance, the EW, $\mathcal{W}^{(\mathrm{II})}_6 = 4.5\mathbb{1} - \sum_{k=1}^{6}\hat{C}^{(\mathrm{II})}_k$, can be used to detect GE, and it tolerates mixing with white noise with $p < 1/3$. When a EW contains all of the operators, i.e., $\mathcal{W}^{(\mathrm{~II})}_8 = 4\mathbb{1} - \hat{M}$ where $\hat{M} = \sum_{k=1}^{8}\hat{C}^{(\mathrm{II})}_k$, it gains a noise tolerance up to $p < 1/2$. It's noticeable that the operator $\hat{M}$ is equivalent to the Bell operator in Mermin-type Bell inequality. As for detections of GE, Mermin-type Bell inequalities involve only the second kind criterion for identifying QC. For many-qubit cases, the Bell operator in Mermin's inequality also contains only the criteria for identifying the correlations between a specific party and others.

Further, we can construct EW to detect states around stabilizing states through criteria of QC, including witnesses for cluster and graph states. If a state is described by stabilizing operators rather than the state vector, we also can derive criteria of QC from these locally measurable operators. For example, $Z_3X_4Z_5$ is one of the stabilizing operators of a five-qubit cluster state [8], and from which we can construct the following operators to specify the QC between the 3$^{\mathrm{rd}}$, the 4$^{\mathrm{th}}$, and the 5$^{\mathrm{th}}$ qubits under the local measurements Z, X, and Z respectively: $\hat{C}'_{0,k} = (\hat{0}_k - \hat{1}_k)(\hat{0}_i\hat{0}_j + \hat{1}_i\hat{1}_j)$, $\hat{C}'_{1,k} = (\hat{1}_k - \hat{0}_k)(\hat{0}_i\hat{1}_j + \hat{1}_i\hat{0}_j)$, for $i,j,k = 3,4,5$, where $\hat{0}_{i(j,k)}$ and $\hat{1}_{i(j,k)}$ have been presented by the eigenstates of corresponding observables. Please note that $Z_3X_4Z_5 = \hat{C}'_{0,k} + \hat{C}'_{1,k}$. For a pure five-qubit cluster state, the expectation values of $\hat{C}'_{0,k}$ and $\hat{C}'_{1,k}$ are both greater than zero [15], then we know there are correlations embedded in this subsystem. By combining and utilizing these correlators which are derived from each stabilizing operator, we can achieve genuine entanglement detections [15].

To show that the proposed scenario can be applied to detect entangled states around a specific state with nonlocal stabilizing operators, let us consider how to construct a witness operator for states around a four-qubit state $\Psi^{(4)}$ [11], which is a superposition of the tensor product of two maximally entangled two-qubit states and a four-qubit GHZ state, $\left|\Psi^{(4)}\right\rangle = \frac{1}{\sqrt{3}}(|0011\rangle_z + |1100\rangle_z - \frac{1}{2}(|0110\rangle_z + |1001\rangle_z + |0101\rangle_z + |1010\rangle_z))$. We formulate eight sets of criteria for identifying QC between a specific party and others. The first type identifications include the following four sets of correlators: $\hat{C}^{(z)}_{0,m} = \hat{0}\hat{0}\hat{1}\hat{1} - X_m(\hat{0}\hat{0}\hat{1}\hat{1})X_m$, and $\hat{C}^{(z)}_{1,m} = \hat{1}\hat{1}\hat{0}\hat{0} - X_m(\hat{1}\hat{1}\hat{0}\hat{0})X_m$, for $m = 1,...,4$. Then, the second type criteria are formulated as: $\hat{C}^{(z)}_{0n,k} = (\hat{0}_{2n+1}\hat{1}_{2n+2} - X_k(\hat{0}_{2n+1}\hat{1}_{2n+2})X_k)(\hat{0}_{2n\oplus 3}\hat{1}_{2n\oplus 4} + \hat{1}_{2n\oplus 3}\hat{0}_{2n\oplus 4})$, and $\hat{C}^{(z)}_{1n,k} = (\hat{1}_{2n+1}\hat{0}_{2n+2} - X_k(\hat{1}_{2n+1}\hat{0}_{2n+2})X_k)(\hat{0}_{2n\oplus 3}\hat{1}_{2n\oplus 4} + \hat{1}_{2n\oplus 3}\hat{0}_{2n\oplus 4})$, where $k = (2n+1), (2n+2)$ for $n = 0,1$; and the symbol "$\oplus$" behaves as the addition of modulo 4 when $n = 1$ and as an ordinary addition when $n = 0$ is met. For invariance of the wave function presented in the eigenbasis of σ_x (σ_y), in analogy, we can construct 8 sets of Hermitian operators, $(\hat{C}^{(x(y))}_{0,m}, \hat{C}^{(x(y))}_{1,m})$ and $(\hat{C}^{(x(y))}_{0n,k}, \hat{C}^{(x(y))}_{1,nk})$, through the replacement of the index z in above Hermitian operators by the index x (y) and constructing the operators in the eigenbasis of $\sigma_{x(y)}$. The expectation values of the above operators are all positive for the state $\Psi^{(4)}$.

Then, we give the following witness operator to detect GE for states close to a $\Psi^{(4)}$ state: $\mathcal{W}_{\Psi^{(4)}} = \tau \mathbb{1} - (\hat{C}^{(x)} + \hat{C}^{(y)} + \hat{C}^{(z)})$, where $\hat{C}^{(i)} = \hat{U}_i^{\otimes 4} \sum_{l=0}^{1} \big(5 \sum_{m=1}^{4} \hat{C}^{(i)}_{l,m} + \sum_{n=0}^{1} \sum_{k=2n+1}^{2n+2} \hat{C}^{(i)}_{ln,k}\big)(\hat{U}_i^{\dagger})^{\otimes 4}$, for $i = x, y, z$, $\hat{U}_x = \hat{H}$, $\hat{U}_y = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}/\sqrt{2}$, $\hat{U}_z = \mathbb{1}$, and $\tau = 36.5$ such that $\mathcal{W}_{\mathrm{GHZ}} - 30\mathcal{W}_{\mathrm{GHZ_p}} > 0$. Moreover, it tolerates mixing with white noise if $p < 15/88 (\approx 0.170455)$. With only three local measurement settings, the above condition for the tolerance of the noise is applicable to a real experiment [7].

Detection of GE for four-level tripartite system.— In order to show further utilities of the proposed scenario, we proceed to provide a witness to detect GE close to a four-level tripartite GHZ state: $|\mathrm{GHZ}_{4\mathrm{x}3}\rangle = 1/2 \sum_{l=0}^{3} |l\rangle_{1,z} \otimes |l\rangle_{2,z} \otimes |l\rangle_{3,z}$. First of all, by a knowledge of the wave function represented in the eigenbasis: $|l\rangle_{j,z}$ for $j = 1, 2, 3$, we have 9 sets of correlators for identifying QC between the m^{th} party and others, and we derive the following operator from the n^{th} set of correlators: $\hat{C}^{(z)}_{mn} = \sum_{k=0}^{3} (\hat{k} - \hat{s}_{kn})_m \hat{k}_p \hat{k}_q$, for $n = 1, ..., 9$; $m, p, q = 1, 2, 3$, and $m \neq p \neq q$; where $\hat{s}_{kn} = \hat{0}, ..., \hat{3}$; $\hat{k} \neq \hat{s}_{kn}$ and $\hat{s}_{kn} \neq \hat{s}_{k'n}$ for $k \neq k'$; and $\hat{C}^{(z)}_{mn} \neq \hat{C}^{(z)}_{mn'}$ for $n \neq n'$.

Secondly, from our knowledge to an alternative representation of a $\mathrm{GHZ}_{4\mathrm{x}3}$ state, $|\mathrm{GHZ}_{4\mathrm{x}3}\rangle = 1/4 \sum_{k,l,r=0}^{3} \delta[(k + l + r) \bmod 4, 0] \, |k\rangle_{1,f} \otimes |l\rangle_{2,f} \otimes |r\rangle_{3,f}$, where $|g\rangle_{j,f} = 1/2 \sum_{h=0}^{3} e^{-i2hg\pi/4} |h\rangle_{j,z}$, we can deduce the following operator from the n^{th} criteria of 9 sets correlators to identify QC between the m^{th} party and others: $\hat{C}^{(f)}_{mn} = \sum_{k=0}^{3} (\hat{F}^{\dagger})^{\otimes 3} (\hat{k} - \hat{s}_{kn})_m \hat{V}_{klr} \hat{F}^{\otimes 3}$, where $\hat{F} = 1/2 \sum_{h,g=0}^{3} e^{i2hg\pi/4} |h\rangle\langle g|$, $\hat{V}_{klr} = \sum_{l,r=0}^{3} \delta[(k + l + r) \bmod 4, 0] \hat{l}_p \hat{r}_q$, and definitions of $\hat{k}$, $\hat{s}_{kn}$, m, p, q, and n are same as the ones mentioned for $\hat{C}^{(z)}_{mn}$.

With the derived correlators, we provide the following EW to detect genuine four-level tripartite entanglement for states close to a $\mathrm{GHZ}_{4\mathrm{x}3}$ state [14]: $\mathcal{W}_{\mathrm{GHZ}_{4\mathrm{x}3}} = 34.131 \mathbb{1} - \sum_{m=1}^{3} \sum_{n=1}^{9} (\hat{C}^{(z)}_{mn} + \hat{C}^{(f)}_{mn})$. Furthermore, when a state mixes with white noise, the EW, $\mathcal{W}_{\mathrm{GHZ}_{4\mathrm{x}3}}$, detects GE if $p < 0.368$. Thus, two local measurement settings are sufficient to detect genuine four-level tripartite entanglement around a $\mathrm{GHZ}_{4\mathrm{x}3}$ state.

Bell type inequality for arbitrary high-dimensional bipartite systems.— Our scenario for deriving Bell type inequality starts with specifications of the criteria for QC. Then, we proceed to verify that any local theory cannot reproduce the correlations embedded in a entangled state. This approach is novel and opposite to the one which has been presented [4].

First, to specify the QC embedded in the maximally entangled state of two d-dimensional parts, $|\psi_d\rangle = 1/\sqrt{d} \sum_{n=0}^{d-1} |n\rangle_{1,z} \otimes |n\rangle_{2,z}$, we represent the wave function in the following eigenbasis: $|l\rangle_{k,j} = 1/\sqrt{d} \sum_{m=0}^{d-1} e^{i2\pi m(l + n_k^{(j)})/d} |m\rangle_{k,z}$, where $n_1^{(1)} = 0$, $n_2^{(1)} = 1/4$, $n_1^{(2)} = 1/2$, and $n_2^{(2)} = -1/4$ correspond to four different local measurements. From our knowledge of the four different representations of the state ψ_d, we give four sets of correlators of QC:

$$\begin{aligned} C^{(12)}_m &= P(v_1^{(1)} = (-m) \bmod d, v_2^{(2)} = m) \\ &\quad - P(v_1^{(1)} = (1 - m) \bmod d, v_2^{(2)} = m), \qquad (5) \\ C^{(21)}_m &= P(v_1^{(2)} = (d - m - 1) \bmod d, v_2^{(1)} = m) \\ &\quad - P(v_1^{(2)} = (-m) \bmod d, v_2^{(1)} = m), \qquad (6) \\ C^{(qq)}_m &= P(v_1^{(q)} = (-m) \bmod d, v_2^{(q)} = m) \\ &\quad - P(v_1^{(q)} = (d - m - 1) \bmod d, v_2^{(q)} = m), \qquad (7) \end{aligned}$$

for $m = 0, 1, ..., d-1$ and $q = 1, 2$. The superscripts, (ij), (i), and (j), indicate that the local measurements $V_1^{(i)}$ and $V_2^{(j)}$ have been selected by the first party and the second one respectively. Thus, we take the summation of all $C^{(ij)}_m$'s,

$$C_d = C^{(11)} + C^{(12)} + C^{(21)} + C^{(22)}, \qquad (8)$$

where $C^{(ij)} = \sum_{m=0}^{d-1} C^{(ij)}_m$, as an identification of the state ψ_d.

For a pure state ψ_d, the correlator $C^{(ij)}_m$ can be evaluated analytically and are given by $C^{(ij)}_m = (\csc^2(\pi/4d) - \csc^2(3\pi/4d))/2d^3$, where $\csc(h)$ is the cosecant of h. Since $C^{(ij)}_m > 0$ for all m's with any finite value of d, we ensure that there are correlations between outcomes of measurements performed on the state ψ_d under four different local measurement settings. Furthermore, we can evaluate the summation of all $C^{(ij)}_m$'s, and then we have $C_{d,\psi_d} = 2(\csc^2(\pi/4d) - \csc^2(3\pi/4d))/d^2$. One can find that C_{d,ψ_d} is an increasing function of d. For instance, if $d = 3$, one has $C_{3,\psi_3} \simeq 2.87293$. In the limit large d, we obtain, $\lim_{d\to\infty} C_{d,\psi_d} = (16/3\pi)^2 \simeq 2.88202$.

We proceed to consider the maximum value of C_d for local hidden variable theories. The following derivation is based on deterministic local models which are specified by *fixing* the outcome of all measurements. This consideration is general since any probabilistic model can be converted into a deterministic one [13]. Substituting a *fixed* set, $(\tilde{v}_1^{(1)}, \tilde{v}_2^{(1)}, \tilde{v}_1^{(2)}, \tilde{v}_2^{(2)})$, into $C^{(ij)}$, C_d turns into $C_{d,\mathrm{LHV}} = \delta[(\tilde{v}_1^{(1)} + \tilde{v}_2^{(1)}) \bmod d, 0] - \delta[-(\tilde{v}_1^{(1)} + \tilde{v}_2^{(1)}) \bmod d, 1] + \delta[(\tilde{v}_1^{(1)} + \tilde{v}_2^{(2)}) \bmod d, 0] - \delta[(\tilde{v}_1^{(1)} + \tilde{v}_2^{(2)}) \bmod d, 1] + \delta[(\tilde{v}_1^{(2)} + \tilde{v}_2^{(2)}) \bmod d, 0] - \delta[-(\tilde{v}_1^{(2)} +$

$\tilde{v}_2^{(2)})\text{mod } d, 1] + \delta[-(\tilde{v}_1^{(2)} + \tilde{v}_2^{(1)})\text{mod } d, 1] - \delta[(\tilde{v}_1^{(2)} + \tilde{v}_2^{(1)})\text{mod} d, 0]$, where $\delta[x, y]$ represent the Kronecker delta symbol. There are three non-vanishing terms at most among the four positive delta functions, and there exist four cases for it, for example, one is that if $\delta[(\tilde{v}_1^{(1)} + \tilde{v}_2^{(1)})\text{mod } d, 0] = \delta[(\tilde{v}_1^{(1)} + \tilde{v}_2^{(2)})\text{mod } d, 0] = \delta[(\tilde{v}_1^{(2)} + \tilde{v}_2^{(2)})\text{mod } d, 0] = 1$ is assigned, we obtain $\tilde{v}_2^{(1)} = \tilde{v}_2^{(2)}$ and then deduce that $\delta[-(\tilde{v}_1^{(2)} + \tilde{v}_2^{(1)})\text{mod } d, 1] = 0$. We also know that there must exist one non-vanishing negative delta function and three vanishing negative ones in the $C_{d,\text{LHV}}$ under the same condition. In the example, the case is $\delta[(\tilde{v}_1^{(2)} + \tilde{v}_2^{(1)})\text{mod } d, 0] = 1$. With these facts, we conclude that $C_{d,\text{LHV}} \leq 2$. One can check other three cases for the four positive delta functions, and then they always result in the same bound. Thus, we realize that $C_{d,\psi_d} > C_{d,\text{LHV}}$ and the QC are stronger than the ones predicted by the local hidden variable theories.

A surprising feature of the new inequality is that the total number of joint probabilities required by each of the presented correlation functions $C^{(ij)}$ is only $2d$, which is much smaller than that in Ref. [5], which is about $O(d^2)$. It implies that the proposed correlation functions only contain the dominant terms to identify correlations. Besides, the proposed scenario is robust against to noise. For instance, if the system is under the condition that $p < 0.30604$, the QC can be maintained for the limit of large d.

Furthermore, although we haven't known yet wether C_d can be utilized to construct EW for detecting arbitrary high-dimensional entanglements around a state ψ_d, as regards the cases have been analyzed, they work for entanglement detection. Take the case for $d = 4$ as an example, the EW, $\mathcal{W}_{\psi_4} = 2.05\mathbb{1} - \hat{C}_4$, where $\hat{C}_4$ is the operator which involves C_4, can be used to detect entanglement for states around ψ_4, and it tolerates mixing with white noise if $p < 0.2881$ [15].

Conclusion.— We have provided a novel and syncretic approach to derive a new Bell type inequality for arbitrary high-dimensional bipartite systems and to construct EW to detect GE around several types of entangled qubits with only a small effort for local measurements. The connection between Mermin-type Bell inequalities and the criteria of QC is discussed. We also show its utility to detect GE around a four-level tripartite GHZ state with two local measurement settings, which help investigation on detections of genuine multilevel and multipartite entanglement in an efficient way.

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[12] If the n^{th} party is uncorrelated with the rest, we recast $P(v_n = k, \text{v} = 3m)$ as $P(v_n = k)P(\text{v} = 3m)$, where $P(v_n = k)$ and $P(\text{v} = 3m)$ denote the probabilities for obtaining results $v_n = k$ and $\text{v} = 3m$ respectively. Then, we have $C_{0,n} = (P(v_n = 0) - P(v_n = 1))P(\text{v} = 0)$ and $C_{1,n} = (P(v_n = 1) - P(v_n = 0))P(\text{v} = 3)$. By the law of conservation of probability, it turns out that $C_{0,n}C_{1,n} \leq 0$. Similarly, if the subsystem composed of the n^{th} party and the m^{th} one is uncorrelated with another, the measured outcomes satisfy $C_{0,nm}C_{1,nm} \leq 0$.
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[14] To prove that $\mathcal{W}_{\text{GHZ}_{4\text{x}3}}$ is an EW to detect GE, firstly, we have to know that $\mathcal{W}_{\text{GHZ}_{4\text{x}3},\text{p}} = \mathbb{1}/4 - |\text{GHZ}_{4\text{x}3}\rangle\langle\text{GHZ}_{4\text{x}3}|$ is a EW which detects GE. It can be shown by the methods presented in Ref. [7]. Secondly, we show that $\mathcal{W}_{\text{GHZ}_{4\text{x}3}}$ satisfies the following criterion to a EW to detect GE: $\mathcal{W}_{\text{GHZ}_{4\text{x}3}} - \gamma\mathcal{W}_{\text{GHZ}_{4\text{x}3},\text{p}} \geq 0$. Our result shows that $\gamma = 28.5$.
[15] Please refer to the forthcoming paper by Che-Ming Li *et al.*.